A classification of spherical conjugacy classes in good characteristic

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Abstract

We classify spherical conjugacy classes in a simple algebraic group over an algebraically closed field of good, odd characteristic.

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Introduction

When studying a transitive action of a group G, it is particularly interesting to understand when a given subgroup B of G acts with finitely many orbits. A particular case of this situation in the theory of algebraic groups is when B is a Borel subgroup of a connected algebraic group G. The G-spaces for which B acts with finitely many orbits are the so-called spherical homogeneous spaces and they include important examples such as the flag variety G/B and symmetric spaces. They are precisely those G-spaces for which the B-action has a dense orbit for the Zariski topology. One may want to understand when homogeneous spaces which are relevant in algebraic Lie theory, such as nilpotent orbits in $\mathrm{Lie}(G)$ and conjugacy classes in G, are spherical. Spherical nilpotent orbits have been classified in [13, 14] when the base field is $\mathbb{C}$ and in [9] when it is an algebraically closed field of good characteristic: they are precisely the orbits of type rA_1 for $r \geq 0$ in the simply-laced case and of type $rA_1 + s\tilde{A}_1$ for $r, s \geq 0$ in the multiply-laced

case. As for conjugacy classes, it is natural to use the interplay with the Bruhat decomposition, which has proven to be a fruitful tool in the past. For instance, it is essential in the description of regular classes ([20]) whose intersection with Bruhat cells is still subject of current research ([7, 8]). This approach has brought to two characterizations of the spherical conjugacy classes in a connected, reductive algebraic group G over an algebraically closed field of zero or good, odd characterisitc ([3, 4, 5]). The first one is given through a formula relating the dimension of a class and the Weyl group element corresponding to a suitable Bruhat cell intersecting the class. The second one states that spherical conjugacy classes are exactly those classes intersecting only Bruhat cells corresponding to involutions in the Weyl group of G. These characterizations can be used in order to give a complete and exhaustive list of the spherical classes of G. In a simple algebraic group over $\mathbb{C}$ they have been classified in [3], making use of the classification of spherical nilpotent orbits. Spherical classes in type G_2 in good characteristic have been classified in [5].

The present paper completes the picture, as it classifies spherical classes in good, odd characteristic. Contrarily to [3], the present work is independent of the classification of spherical unipotent conjugacy classes existing in the literature. Since Springer corrispondence holds in good characteristic, it provides an elementary classification of spherical nilpotent orbits alternative to [9], where Kempf-Russeau theory is involved and where the aid of a computer programme is needed to deal with the exceptional types. The crucial tools in our method are just those conditions in the characterizations in [3, 4, 5] whose proofs are general and rather short. The arguments used for this classification can also be transferred to the characteristic zero situation, providing an alternative, elementary approach to [13, 14], although by case-by-case considerations.

After fixing notations and recalling basic notions in §1, we introduce spherical conjugacy classes and their characterizations in §2. We proceed providing the list of spherical conjugacy classes through a case-by-case analysis in §3.

The result is as when the base field is $\mathbb{C}$: In the simply-laced case spherical conjugacy classes are either semisimple or unipotent and the semisimple ones are all symmetric spaces if G is simply-connected. In type G_2 spherical conjugacy classes are again either semisimple or unipotent but, as in types B_n and C_n, there are spherical semisimple classes that are not symmetric. Just as in other situations involving spherical homogeneous spaces (for example, in the description of maximal spherical ideals of Borel subalgebras [15]) the doubly-laced case is slightly more involved. The new phenomenon in the present situation is that there appear spherical classes that are neither semisimple nor unipotent.

1 Notation

Unless otherwise stated G will denote a simple algebraic group over an algebraically closed field k of good odd characteristic ([19, §I.4]). When we consider an integer as an element in k we shall mean its image in the prime field of k. We shall denote by Φ the root system relative to a fixed Borel subgroup B and a maximal torus T of G; by $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the corresponding set of simple roots and by Φ^+ the set of positive roots. We shall use the numbering of the simple roots in [1, Planches I-IX]. The highest positive root will be denoted by β_1. For generators of T we shall adopt the notation in [21, Lemma 19]. We will put $W = N(T)/T$ and s_α will indicate the reflection corresponding to the root α. Given an element $w \in W$ we shall denote by $\dot{w}$ a representative of w in $N(T)$. The maximal unipotent subgroup of B will be denoted by U. For a root α the elements of the associated root subgroup X_α will be denoted by $x_\alpha(t)$. We shall frequently use that $x_{-\alpha}(t) \in Bs_\alpha B$ and that if $\alpha \in \Delta$ and $w\alpha \in \Phi^+$ then $BwBs_\alpha B = Bws_\alpha B$ whereas if $w\alpha \in -\Phi^+$ then $BwBs_\alpha B \subset BwB \cup Bws_\alpha B$ ([18, Lemma 8.3.7]).

Given an element $x \in G$ we shall denote by $\mathcal{O}_x$ the conjugacy class of x in G and by H_x the centralizer of x in $H \leq G$. For the dimension of unipotent conjugacy classes in arbitrary good characteristic we refer to [6, Chapter 13] and [16, Theorem 2.6].

2 Characterizations through the Bruhat decomposition

In this section we shall introduce the characterizations of spherical conjugacy classes that we will use in the sequel.

Definition 2.1 *Let G be a connected reductive algebraic group. A homogeneous space G/H is* spherical *if it has a dense orbit for some Borel subgroup of G.*

By an abuse of notation we shall say that $g \in G$ is spherical if its class $\mathcal{O}_g \simeq G/G_g$ is so.
For a conjugacy class $\mathcal{O}$ in G, we shall denote by $\mathcal{V}$ the set of B-orbits in $\mathcal{O}$. It is well-known ([2, 22] in characteristic 0, [10, 12] in positive characteristic) that $\mathcal{O}$ is a spherical conjugacy class if and only if $\mathcal{V}$ is finite.

Since $G = \bigcup_{w \in W} BwB$, for every class $\mathcal{O}$ there is a natural map $\phi\colon \mathcal{V} \to W$ associating to $v \in \mathcal{V}$ the element w in the Weyl group of G for which $v \subset BwB$.

Besides, there is a unique $w \in W$ for which $BwB \cap \mathcal{O}$ is dense in $\mathcal{O}$ and this element is maximal in $\mathrm{Im}(\phi)$ with respect to the Bruhat ordering ([3, p. 32]). We shall denote such an element by $w_{\mathcal{O}}$.

There are two characterizations of spherical classes in G. Let ℓ denote the usual length function on W and let $\mathrm{rk}(1-w)$ denote the rank of the operator $1-w$ in the geometric representation of W.

Theorem 2.2 *([3, Theorem 25], [4, Theorem 4.4]) A class $\mathcal{O}$ in a connected reductive algebraic group G over an algebraically closed field of zero or good odd characteristic is spherical if and only if there exists v in $\mathcal{V}$ such that $\ell(\phi(v)) + \mathrm{rk}(1-\phi(v)) = \dim \mathcal{O}$. If this is the case, v is the dense B-orbit and $\phi(v) = w_{\mathcal{O}}$.*

Theorem 2.3 *([4, Theorem 2.7],[5, Theorem 5.7]) A class $\mathcal{O}$ in a connected reductive algebraic group G over an algebraically closed field of zero or odd, good characteristic is spherical if and only if $\mathrm{Im}(\phi)$ contains only involutions in W.*

Remark 2.4 Regular classes in a reductive algebraic group whose semisimple quotient is not of type rA_1 cannot be spherical. Indeed, by [20, Theorem 8.1], regular classes intersect Bruhat cells corresponding to Coxeter elements.

Remark 2.5 Let g and x be elements in G with $G_x = G_g$. Then $\mathcal{O}_g \simeq G/G_g$ is spherical if and only if $\mathcal{O}_x \simeq G/G_x$ is so. In particular, if $g^2 \in Z(G)$ then $\mathcal{O}_x$ is a symmetric space, hence spherical by [17, Corollary 4.3].

Let $g \in B$ with Jordan decomposition $g = su \in TU$. Then $u \in G_s^\circ$ and $G_g = G_s \cap G_u$. Therefore, if BG_g is dense in G then BG_s and BG_u are dense in G. In other words, if $\mathcal{O}_g$ is spherical then $\mathcal{O}_s$ and $\mathcal{O}_u$ are also spherical. We can refine this argument.

Lemma 2.6 *Let $g \in B$ with Jordan decomposition $g = su \in TU$. If $\mathcal{O}_g$ is spherical then $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical in G and the class $\mathcal{O}'_u$ of u in G_s° is spherical in G_s°.*

Proof. The subgroup $B_1 = B \cap G_s^\circ$ is a Borel subgroup for G_s° containing T. If $x = b_1\dot{w}b_2 \in B_1(N(T) \cap G_s^\circ)B_1$ is a Bruhat decomposition of $x \in G_s^\circ$ then it is also a Bruhat decomposition of x in G. If $\mathrm{v} \in \mathcal{O}'_u$ with $\mathrm{v} = b_1\dot{w}b_2$ then $s\mathrm{v} = sb_1\dot{w}b_2 \in BN(T)B \cap \mathcal{O}_g$ forcing $\dot{w}^2 \in T$ by Theorem 2.3 applied to G. By Theorem 2.3 applied to G_s° we deduce that $\mathcal{O}'_u$ is spherical therein. $\square$

3 The classification

We aim at a classification of spherical conjugacy classes in good odd characteristic since the classification for $k = \mathbb{C}$ in [3] holds for every algebraically closed field of characteristic zero. The property of being spherical for G/H depends only on $\mathrm{Lie}(G)$ and $\mathrm{Lie}(H)$ so the classification will not distinguish groups isogenous to G and it will be up to a central element in G. The main tools will be the sufficient condition in Theorem 2.2 ([3, Theorem 5]) and the necessary condition in Theorem 2.3 ([4, Theorem 2.7]). If G is of type G_2 the classification in good characteristic is given in [5, §2.1] and we provide it here for the sake of completeness.

3.1 Type G_2

Theorem 3.1 *Let G be of type G_2. The spherical classes are either semisimple or unipotent. The semisimple ones are represented by $h_{\alpha_1}(-1)$ and $h_{\alpha_1}(\zeta)$ for ζ a fixed primitive third root of 1. The unipotent ones are those of type A_1 and $\tilde{A}_1$.*

3.2 Type A_n

Theorem 3.2 *Let $G = SL_{n+1}(k)$. If $n = 1$ all classes are spherical. If $n \geq 2$ the spherical classes are either semisimple or unipotent. The semisimple ones are those corresponding to matrices with at most two distinct eigenvalues and they are all symmetric spaces. The unipotent ones are those associated with partitions of type $(2^m, 1^{n+1-2m})$ for $m = 1, \ldots, [\frac{n+1}{2}]$.*

Proof. If $n = 1$ all Bruhat cells correspond to involutions in W so every class is spherical by Theorem 2.3.

Let G be $SL_{n+1}(k)$ with $n \geq 2$, let B, T, U be as usual and let $\mathcal{O} = \mathcal{O}_u$ be a unipotent class. By Jordan theory we may assume that $u = x_{-\alpha_1}(c_1) \cdots x_{-\alpha_n}(c_n)$ with $c_i \in \{0, 1\}$. By [18, Lemma 8.1.4(i), Lemma 8.3.7] this element lies in $Bs_1^{c_1} \cdots s_n^{c_n} B$. This cell corresponds to an involution only if $c_i c_{i+1} = 0$ for all $i = 1, \ldots, n-1$. Theorem 2.3 implies that if $\mathcal{O}_u$ is spherical its associated partition is of type $(2^m, 1^{n+1-2m})$. Conversely, let $\mathcal{O}$ be a unipotent class corresponding to $(2^j, 1^{n+1-2j})$, with $2j \leq n+1$. Let $\beta_i = \alpha_i + \cdots + \alpha_{n-i+1}$ for $i = 1, \ldots, j$. The element $x_{-\beta_1}(1) \cdots x_{-\beta_j}(1)$ lies in $\mathcal{O}$ and its B-orbit satisfies the condition in Theorem 2.2 so $\mathcal{O}$ is spherical.

Let $s = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots, \lambda_l I_{n_l}) \in T$. If l is greater than 2 then s is conjugate to some matrix $t = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, t_1)$ for some invertible diagonal

submatrix t_1. Then t lies in the reductive subgroup $H = \langle T, X_{\pm\alpha_1}, X_{\pm\alpha_2}\rangle$ and it is regular therein. Besides, $B_1 = B \cap H$ is a Borel subgroup of H containing T. By Remark 2.4 some conjugate of t lies in a Bruhat cell of H that does not correspond to an involution. Therefore, if $\mathcal{O}_s$ is spherical semisimple, s has at most 2 eigenvalues. Conversely, suppose that $s \in T$ has 2 eigenvalues. We may assume $s = \mathrm{diag}(\lambda I_m, \mu I_{n+1-m})$. Let ζ be a primitive $2(n+1)$-th root of unity if $n+1-m$ is odd and let $\zeta = 1$ if $n+1-m$ is even. Let also $s_0 = \mathrm{diag}(\zeta I_m, -\zeta I_{n+1-m})$. Then $s_0^2 \in Z(G)$ and $G_s = G_{s_0}$. By Remark 2.5 the class $\mathcal{O}_s$ is symmetric.

We will now show that there is no spherical element with Jordan decomposition $x = su$ with $s \notin Z(G)$ and $u \neq 1$. Were this the case, we could assume that $s = \mathrm{diag}(\lambda I_m, \mu I_{n+1-m})$ with $m \geq 2$ and that x is conjugate to some $y = \begin{pmatrix} \lambda u_1 & & & \\ & \lambda & & \\ & 1 & \lambda & \\ & & & \mu u_2 \end{pmatrix}$ with $u_1 \in X_{\alpha_1} \cdots X_{\alpha_{m-3}}$ and $u_2 \in X_{\alpha_{m+1}} \cdots X_{\alpha_n}$. Then $y' = \begin{pmatrix} \lambda u_1 & & & \\ & \lambda & & \\ & 1 & \lambda & \\ & & 1 & \mu u_2 \end{pmatrix} \in \mathcal{O}_x \cap B s_{m-1} s_m B$. By Theorem 2.3 we have the statement. $\square$

3.3 Type C_n

Theorem 3.3 *Let $G = Sp_{2n}(k)$ for $n \geq 2$. The spherical semisimple classes are represented up to a sign by $\sigma_l = \mathrm{diag}(-I_l, I_{n-l}, -I_l, I_{n-l})$ for $l = 1, \ldots, \left[\frac{n}{2}\right]$; $a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ and $c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$ for $\lambda^2 \neq 0, 1$. The unipotent ones are those whose associated partition is of type $(2^m, 1^{2n-2m})$ for $m = 1, \ldots, n$. The spherical classes that are neither semisimple nor unipotent are represented up to a sign by the elements $\sigma_l u$ where $u \in G_{\sigma_l} \cong Sp_{2l}(k) \times Sp_{2n-2l}(k)$ is unipotent and corresponds to the partition $(2, 1^{2n-2})$.*

Proof. We view G as the subgroup of $GL_{2n}(k)$ of matrices preserving the bilinear form associated with the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with respect to the canonical basis of k^{2n}. We choose B as the subgroup of G of matrices of the form $\begin{pmatrix} A & AX \\ 0 & {}^tA^{-1} \end{pmatrix}$ where A is an invertible upper triangular matrix, ${}^tA^{-1}$ is its inverse transpose and X is a symmetric matrix. The torus T is the subgroup of diagonal matrices in B.

Let $s \in T$. If s has at least 4 eigenvalues then it is either conjugate to a matrix of type $s' = \mathrm{diag}(r, \lambda, \mu, r^{-1}, \lambda^{-1}, \mu^{-1})$ for some $\lambda,\, \mu \in k$ with $\lambda \notin \{\lambda^{-1}, \mu, \mu^{-1}\}$ and $\mu^2 \neq 1$ and some invertible diagonal submatrix r, or to a matrix of type $s'' = \mathrm{diag}(\lambda, 1, -1, t, \lambda^{-1}, 1, -1, t^{-1})$ for some invertible diagonal submatrix t and some $\lambda \in k$ with $\lambda^2 \neq 1$. The element s' is regular in $H = \langle T,\, X_{\pm\alpha_{n-1}},\, X_{\pm\alpha_n}\rangle$ whereas s'' is regular in $H' = \langle T,\, X_{\pm\alpha_1},\, X_{\pm\alpha_2}\rangle$. By Remark 2.4 applied to H and H' the class $\mathcal{O}_s$ cannot be spherical.

Let now s have exactly 3 eigenvalues. Then one of these is ± 1 and it is not restrictive to assume that they are $\lambda, \lambda^{-1}, 1$ with $\lambda^2 \neq 1$. If the multiplicity of $\lambda^{\pm 1}$ is greater than 1, s is conjugate to some $r' = \mathrm{diag}(\lambda, \lambda^{-1}, 1, r_1, \lambda^{-1}, \lambda, 1, r_1^{-1})$ with r_1 an invertible, diagonal submatrix. The element r' lies and is regular in the subgroup H' described above. By Remark 2.4 the class $\mathcal{O}_s$ cannot be spherical. On the other hand if s has exactly 3 eigenvalues $\lambda^{\pm 1}, 1$ (up to a sign) with the multiplicity of $\lambda^{\pm 1}$ equal to 1, then $\mathcal{O}_s$ is spherical. Indeed, the representative of its class in [3, Theorem 15] works also in odd characteristic and its B-orbit satisfies the condition of Theorem 2.2.

Let $s \in T$ have exactly 2 eigenvalues. They are either $1,\, -1$ so that $\mathcal{O}_s$ is symmetric, or λ, λ^{-1} for $\lambda^2 \neq 1$. In the latter case s is conjugate to $\mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ and its centralizer coincides with the centralizer of $s_0 = \mathrm{diag}(\zeta I_n, -\zeta I_n)$ for ζ a primitive fourth root of 1. Since $\mathcal{O}_{s_0}$ is a symmetric space, $\mathcal{O}_s$ is spherical by Remark 2.5.

Unipotent classes in G are parametrized through Jordan theory by partitions where odd terms occur pairwise. Let $\mathcal{O}_u$ be a unipotent class and let $\underline{\lambda}$ be its associated partition. Let $\underline{\mu} = (\mu_1, \ldots, \mu_l)$ be obtained by taking a representative of each term occurring pairwise in $\underline{\lambda}$ and let $\underline{\nu} = (\nu_1, \ldots, \nu_m)$ be obtained by taking the (even) terms without repetition in $\underline{\lambda}$, so that $2n = |\underline{\nu}| + 2|\underline{\mu}|$. A representative u' of $\mathcal{O}_u$ can be taken in the subgroup isomorphic to $Sp_{2\mu_1}(k) \times \cdots \times Sp_{2\mu_l}(k) \times Sp_{\nu_1}(k) \times \cdots \times Sp_{\nu_m}(k)$ obtained by repeating the immersion of $Sp_{2d_1}(k) \times Sp_{2d_2}(k)$ into $Sp_{2(d_1+d_2)}(k)$ given by $\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} A_1 & & B_1 & \\ & A_2 & & B_2 \\ C_1 & & D_1 & \\ & C_2 & & D_2 \end{pmatrix}$. The component of u' in $Sp_{\nu_j}(k)$ corresponds to the partition (ν_j). Besides, the component of u' in $Sp_{2\mu_i}(k)$ can be taken to lie and be regular in the subgroup isomorphic to $SL_{\mu_i}(k)$ obtained through the immersion $M \mapsto \mathrm{diag}(M, {}^tM^{-1})$. In particular, the choices of Borel subgroups we made are

all compatible and u' is regular in a subgroup of type $A_{\mu_1-1}\times\cdots\times A_{\mu_l-1}\times C_{\nu_1}\times\cdots\times C_{\nu_m}$. Therefore, if u is spherical, we necessarily have $\mu_i=2$ and $\nu_j=1$ for every i and j. Conversely, let $\underline{\lambda}=(2^j,1^{2n-2j})$ and let $\mathcal{O}_j$ be the unipotent class associated with $\underline{\lambda}$. Then for $\beta_q=2\alpha_q+\cdots+2\alpha_{n-1}+\alpha_n$ for $q=1,\ldots,n$ the B-orbit of the element $x_{-\beta_1}(1)\cdots x_{-\beta_j}(1)$ satisfies the condition in Theorem 2.2 for $\mathcal{O}_j$ (cfr. [3, Theorem 12]).

Let $g=su$ be the Jordan decomposition of a spherical element in G with $s\notin Z(G)$ and $u\neq 1$. Then $\mathcal{O}_s$ is spherical so we may assume $s=a_\lambda, c_\lambda$ or σ_l. The case $s=a_\lambda$ is ruled out because we would have $\dim O_{a_\lambda u}>\dim\mathcal{O}_{a_\lambda}=\dim B$. Let us assume that $s=c_\lambda$. Then $u\in G_s\cong k^*\times Sp_{2n-2}(k)$ and it is spherical therein so it corresponds to a partition of the form $(2^m,1^{2n-2-2m})$. The class $\mathcal{O}_{c_\lambda u}$ may be represented by a matrix of the form $\begin{pmatrix} A & \\ {}^tA^{-1}X & {}^tA^{-1}\end{pmatrix}$ where $A=\begin{pmatrix}\lambda & & \\ 1 & 1 & \\ & & I_{n-2}\end{pmatrix}$ and $X=\mathrm{diag}(0,I_m,0_{n-m-1})$. Such a representative is contained in $TX_{-\alpha_1}X_{-\beta_2}\cdots X_{-\beta_{m+1}}$ with notation as above, and it lies in $Bs_1s_{\beta_2}\cdots s_{\beta_{m+1}}B$. By Theorem 2.3 this is not possible. It follows that we necessarily have $s=\sigma_l$ for some l.

In this case G_s is a subgroup isomorphic to $Sp_{2l}(k)\times Sp_{2n-2l}(k)$ and $u=(u_1,u_2)$ lies in G_s and it is spherical therein. Then u_1 and u_2 are spherical in the respective components. We claim that u_1 and u_2 correspond to partitions with no repeated terms. If $\underline{\lambda}=(2,2,\underline{\lambda}')$ were the partition associated with u_2 then $\mathcal{O}_u$ would contain a matrix of the form

$$\begin{pmatrix}
A_1 & & & & A_1X_1 & & & & \\
& 1 & 1 & & & & & & \\
& & 1 & 1 & & & & & \\
& & & A_2 & & & & & A_2X_2 \\
& & & & {}^tA_1^{-1} & & & & \\
& & & & & 1 & & & \\
& & & & & 1 & -1 & & \\
& & & & & & 1 & & \\
& & & & & & & & {}^tA_2^{-1}
\end{pmatrix}$$

where $\begin{pmatrix}A_1 & A_1X_1\\ & {}^tA_1^{-1}\end{pmatrix}$ represents $-u_1$, $\begin{pmatrix}A_2 & A_2X_2\\ & {}^tA_2^{-1}\end{pmatrix}$ represents a unipotent element corresponding to $\underline{\lambda}'$ and A_1, A_2 are upper triangular. Such a representative lies in $Bs_ls_{l+1}B$ leading to a contradiction. The case in which the partition associated with u_1 has repeated terms can be treated similarly. We claim that u can have only one nontrivial component. If this were not the case, the

class would be represented by a matrix of the form $\begin{pmatrix} A & \\ {}^tA^{-1}X & {}^tA^{-1} \end{pmatrix}$ where $A = \begin{pmatrix} -I_{l-1} & & & \\ & -1 & & \\ & 1 & 1 & \\ & & & I_{n-l-1} \end{pmatrix}$ and $X = \mathrm{diag}(0_l, 1, 0_{n-l-1})$. Such a representative lies in $TX_{-\alpha_l}X_{-\beta_{l+1}}$ and its Bruhat cell corresponds to $s_l s_{\beta_{l+1}}$ which is not an involution. Conversely, for all classes of type $\sigma_l u$ with $u \in G_{\sigma_l}$ corresponding to the partition $(2, 1^{2n-2})$ the representative in [3, Theorem 21] is defined in odd characteristic and its B-orbit satisfies the condition of Theorem 2.2. $\square$

3.4 Type D_n

Theorem 3.4 *Let $G = SO_{2n}(k)$ for $n \geq 4$. The spherical classes in G are either semisimple or unipotent. The semisimple ones are represented, up to a central element and up to the automorphism arising from an automorphism of the Dynkin diagram, by $\sigma_l = \mathrm{diag}(-I_l, I_{n-l}, -I_l, I_{n-l})$ for $l = 1, \ldots, \left[\frac{n}{2}\right]$; $a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ and $c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$ for $\lambda^2 \neq 0, 1$. The unipotent ones are those associated with the partitions $(2^{2m}, 1^{2n-4m})$ for $m = 1, \ldots, \left[\frac{n}{2}\right]$ and $(3, 2^{2m}, 1^{2n-3-4m})$ for $m = 1, \ldots, \left[\frac{n}{2}\right] - 1$.*

Proof. We view $SO_{2n}(k)$ as the subgroup of $SL_{2n}(k)$ of matrices preserving the bilinear form associated with the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ with respect to the canonical basis of k^{2n}. We choose B as the subgroup of G of matrices of the form $\begin{pmatrix} A & AX \\ 0 & {}^tA^{-1} \end{pmatrix}$ where A is an invertible upper triangular matrix, ${}^tA^{-1}$ is its inverse transpose and X is a skew symmetric matrix. We fix $T \subset B$ as its subgoup of diagonal matrices.

Let $s \in T$. If s has at least 4 eigenvalues, up to a Dynkin diagram automorphism s is conjugate to some $r = \mathrm{diag}(\lambda, \mu, \nu, t, \lambda^{-1}, \mu^{-1}, \nu^{-1}, t^{-1})$ for a suitable invertible diagonal submatrix t and some scalars λ, μ, ν, with ν possibly equal to λ^{-1}. Such a matrix is regular in $\langle T, X_{\pm\alpha_1}, X_{\pm\alpha_2} \rangle$ so $\mathcal{O}_s$ is not spherical by Remark 2.4.

Let us now assume that s has exactly 3 eigenvalues, say, $\lambda, \lambda^{-1}, 1$ and that the multiplicity of λ is greater than 1. Then $\mathcal{O}_s$ contains a matrix of the form r as above with $\nu = \lambda^{-1}$ and $\mu = 1$ and the above argument shows that $\mathcal{O}_s$ cannot be spherical. On the other hand, if $s \in T$ has exactly 3 eigenvalues $\lambda^{\pm 1}, 1$

with the multiplicity of λ equal to 1, up to an automorphism of the Dynkin diagram, s is conjugate to a matrix of the form $c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$. Its centralizer is equal to the identity component of the centralizer of the involution $\sigma_1 = \mathrm{diag}(-1, I_{n-1}, -1, I_{n-1})$. Since G/H is spherical if and only if G/H° is so, $\mathcal{O}_{c_\lambda}$ is spherical.

Let now s have exactly two eigenvalues. If they are ± 1 then $s^2 = 1$ and $\mathcal{O}_s$ is symmetric. If the eigenvalues are $\lambda^{\pm 1}$ with $\lambda^2 \neq 1$ we may assume that s is conjugate to $a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ whose centralizer is independent of λ in the given range. Since for ζ a primitive fourth root of 1 we have $a_\zeta^2 \in Z(G)$, by Remark 2.5 all those classes are symmetric spaces, hence spherical.

Let us now consider unipotent classes in G. There might be two classes in G associated by Jordan theory with the same partition. However, two such classes are mapped onto each other by a group automorphism araising from the automorphism of the Dynkin diagram. Thus, one is spherical if and only if the other is so and we will not need to distinguish them.

Let u be a unipotent element in G. It is well-known that the even terms in its associated partition $\underline{\lambda}$ occur pairwise. Let $\underline{\mu} = (\mu_1, \ldots, \mu_l)$ be obtained taking a representative of each term occurring pairwise in $\underline{\lambda}$ and let $\underline{\nu} = (\nu_1, \ldots, \nu_m)$ be obtained by taking the remaining distinct odd terms so that $2n = 2|\underline{\mu}| + |\underline{\nu}|$. A representative u' of $\mathcal{O}_u$ can be taken in the subgroup isomorphic to $SO_{\nu_1+\nu_2}(k) \times \cdots \times SO_{\nu_{m-1}+\nu_m}(k) \times SO_{2\mu_1}(k) \times \cdots \times SO_{2\mu_l}(k)$ obtained by repeatedly immerging $SO_{2d_1}(k) \times SO_{2d_2}(k)$ into $SO_{2(d_1+d_2)}(k)$ by $\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} A_1 & & B_1 & \\ & A_2 & & B_2 \\ C_1 & & D_1 & \\ & C_2 & & D_2 \end{pmatrix}$. The component of u' in $SO_{\nu_i+\nu_{i+1}}(k)$ is associated with the partition (ν_i, ν_{i+1}). The component of u' in $SO_{2\mu_i}(k)$ can be chosen to lie and be regular in the subgroup isomorphic to $SL_{\mu_i}(k)$ obtained through the immersion $M \mapsto \mathrm{diag}(M, {}^tM^{-1})$. As in type C_n this forces $\mu_i \leq 2$ for all i. We shall now show that $\nu_1 \leq 3$. Let $\nu_1 = 2l + 1$ and $\nu_2 = 2j - 1$ with $l \geq j$. For the component x of u' in $SO_{\nu_1+\nu_2}(k)$ we take $x = \begin{pmatrix} A & \\ AX & {}^tA^{-1} \end{pmatrix}$ where $A = \begin{pmatrix} 1 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix}$ and $AX = \begin{pmatrix} 0_{l-1} & & & \\ & 1 & 1 & \\ & -1 & 0 & \\ & & & 0_{j-1} \end{pmatrix}$. Let us put

$\gamma_l = \alpha_l + 2(\alpha_{l+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$. Then the image of x in G lies in $X_{-\alpha_1} \cdots X_{-\alpha_{l+j-1}} X_{-\gamma_l} B$ and it lies in the union of the cells corresponding to $s_1 \cdots s_{l-1} s_{l+1} s_{l+2} \cdots s_{l+j-1} s_{\gamma_l}$, $s_1 \cdots s_l s_{l+2} s_{l+3} \cdots s_{l+j-1} s_{\gamma_l}$ and $s_1 \cdots s_{l+j-1} s_{\gamma_l}$. Since none of the involved elements of W is an involution unless $l + j \leq 2$ we conclude that $\nu_1 \leq 3$.

Conversely, let $\mathcal{O}_u$ be a unipotent class corresponding to $(2^{2m}, 1^{2n-4m})$ or to $(3, 2^{2m}, 1^{2n-3-4m})$. The matrices in [3, Theorem 12] represent these classes also when $\mathrm{char}(k)$ is odd and their B-orbits satisfy the condition in Theorem 2.2.

We will show that there is no spherical element with Jordan decomposition $g = su$ with $s \notin Z(G)$ and $u \neq 1$. We might assume that $s = c_\lambda$ or $\sigma_l = \mathrm{diag}(-I_l, I_{n-l}, -I_l, I_{n-l})$ because $\dim B = \dim \mathcal{O}_{a_\lambda} < \dim \mathcal{O}_{a_\lambda u}$.

The subgroup $G^\circ_{c_\lambda}$ is of type $D_{n-1} \times k^*$ and it is generated by T and the root subgroups $X_{\pm\alpha_2}, \ldots, X_{\pm\alpha_n}$. If $u \in G^\circ_{c_\lambda}$ corresponds to a partition $\underline{\mu}$ of $2n-2$ with repeated terms, then $\mu = (2, 2, \underline{\mu}')$ for some partition $\underline{\mu}'$ of $2n-4$ and $c_\lambda u$ would be conjugate to a matrix of the form
$$\begin{pmatrix}
\lambda &&&&&&& \\
1 & 1 &&&&&& \\
& 1 & 1 &&&&& \\
&&& A_1 &&&& A_1X_1 \\
&&&& \lambda^{-1} & -\lambda^{-1} && \\
&&&&& 1 & -1 & \\
&&&&&& 1 & \\
&&&&&&& {}^tA_1^{-1}
\end{pmatrix}$$
where $\begin{pmatrix} A_1 & A_1X_1 \\ & {}^tA_1^{-1} \end{pmatrix}$ represents a unipotent element associated with $\underline{\mu}'$ and A_1 is upper triangular. Such a representative lies in Bs_1s_2B. Thus, u has no repeated terms and it must be associated with $(3, 1^{2n-5})$. Then su is conjugate to some $x = \begin{pmatrix} A & \\ AX & {}^tA^{-1} \end{pmatrix}$ where $A = \begin{pmatrix} \lambda &&& \\ 1 & 1 && \\ & 1 & 1 & \\ &&& I_{n-3} \end{pmatrix}$ and $AX = \begin{pmatrix} 0 &&& \\ & 1 & 1 & \\ & -1 & 0 & \\ &&& 0_{n-3} \end{pmatrix}$ so $x \in X_{-\alpha_1} X_{-\alpha_2} X_{-(\alpha_2+2\alpha_3+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n)}$ and it lies in the cell corresponding to $s_1 s_2 s_{\alpha_2+2\alpha_3+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}$ which is not an involution. Thus, $s = \sigma_l$ for some l. The argument used for G of type C_n can be adapted in order to show that the partition associated with any component of $u = (u_1, u_2) \in SO_{2l}(k) \times SO_{2n-2l}(k) = G^\circ_{\sigma_l}$ cannot have repeated terms of size greater than 1. It follows that such components can only be associated with $(3, 1, \ldots, 1)$. Suppose that u_2 is nontrivial so that $n \geq l + 2$. We may find a representative x of $\mathcal{O}_g$ as fol-

lows $x = \begin{pmatrix} A_1 & & & A_1X_1 & & & & \\ 1 & 1 & & & & & & \\ & 1 & 1 & & & & & \\ & & & I_{n-l-2} & & & & \\ & & & & {}^tA_1^{-1} & 1 & & \\ & 1 & 1 & & & 1 & -1 & \\ & -1 & 0 & & & & 1 & \\ & & & & & & & I_{n-l-2} \end{pmatrix}$ where $\begin{pmatrix} A_1 & A_1X_1 \\ & {}^tA_1^{-1} \end{pmatrix}$ represents u_1 and A_1 is an upper triangular invertible matrix. Then x lies in $TX_{-\alpha_l}X_{-\alpha_{l+1}}X_{-(\alpha_{l+1}+2\alpha_{l+2}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n)}$ and in the cell corresponding to $s_ls_{l+1}s_{\alpha_{l+1}+2\alpha_{l+2}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}$ leading to a contradiction. Similarly, we may exclude that u_1 corresponds to $(3,1,\ldots,1)$, concluding the proof. $\square$

3.5 Type B_n

Theorem 3.5 *Let $G = SO_{2n+1}(k)$. The spherical semisimple classes in G are represented by $\rho_l = \mathrm{diag}(1,-I_l,I_{n-l},-I_l,I_{n-l})$ for $l = 1,\ldots,n$; by $d_\lambda = \mathrm{diag}(1,\lambda,I_{n-1},\lambda^{-1},I_{n-1})$ and by $b_\lambda = \mathrm{diag}(1,\lambda I_n,\lambda^{-1}I_n)$ with $\lambda^2 \neq 0,1$. The unipotent ones are those associated with $(2^{2m},1^{2n+1-4m})$ for $m = 1,\ldots,[\frac{n}{2}]$ and $(3,2^{2m},1^{2n-2-4m})$ for $m = 1,\ldots,[\frac{n-1}{2}]$. The spherical classes that are neither semisimple nor unipotent are represented by $\rho_n u$ where $u \in G^\circ_{\rho_n} \cong SO_{2n}(k)$ is a unipotent element associated with $(2^{2m},1^{2n-4m})$ for $m = 1,\ldots,[\frac{n}{2}]$.*

Proof. We view G as the subgroup of $SL_{2n+1}(k)$ of matrices preserving the bilinear form associated with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$ with respect to the canonical basis of k^{2n+1}. We fix B to be the subgroup of matrices of the form $\begin{pmatrix} 1 & 0 & {}^t\gamma \\ -A\gamma & A & AX \\ 0 & 0 & {}^tA^{-1} \end{pmatrix}$ where A is an invertible upper triangular matrix, γ is a column in k^n and the symmetric part of X is $-\frac{1}{2}\gamma{}^t\gamma$. We fix $T \in B$ as its subgroup of diagonal matrices.

Let $s \in T$ be a spherical element in G. Adapting the proof of Theorem 3.4 by using the embedding ι of $SO_{2n}(k)$ into $SO_{2n+1}(k)$ given by $X \mapsto \begin{pmatrix} 1 & \\ & X \end{pmatrix}$ we see that s has at most 4 eigenvalues and one of them is 1. Moreover, if they were 4, we could take $s = \mathrm{diag}(1,t,\lambda,-1,t^{-1},\lambda^{-1},-1)$ for some invertible diagonal submatrix t and some scalar λ. Then s would be regular in $\langle T,\ X_{\pm\alpha_{n-1}},\ X_{\pm\alpha_n}\rangle$ so

by Remark 2.4 this case may not occur. Therefore s has 2 or 3 eigenvalues. If s has 2 eigenvalues it is conjugate to the involution $\rho_l = \mathrm{diag}(1, -I_l, I_{n-l}, -I_l, I_{n-l})$ for some l. If s has 3 eigenvalues they are $1, \lambda, \lambda^{-1}$. Then the multiplicity of λ and 1 cannot be both greater than 1 as one can see using the embedding ι and the element $\iota(r')$ adapting the discussion in Theorem 3.4. Thus, s is conjugate either to $b_\lambda = \mathrm{diag}(1, \lambda I_n, \lambda^{-1} I_n)$ or to $d_\lambda = \mathrm{diag}(1, \lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$, for $\lambda^2 \neq 1, 0$. A representative of $\mathcal{O}_{b_\lambda}$ satisfying the condition in Theorem 2.2 is to be found in [3, Theorem 15] and it is well-defined in odd characteristic, too. Moreover, $G_{d_\lambda} = G^\circ_{\rho_1}$ hence b_λ and d_λ are indeed spherical.

Partitions where even terms occur pairwise parametrize unipotent conjugacy classes in G. Let u be a spherical unipotent element in G associated with the partition $\underline{\lambda}$. Let $\underline{\mu}$ and $\underline{\nu}$ be constructed as in Theorem 3.4 with $2n+1 = 2|\underline{\mu}| + |\underline{\nu}|$. We may find a representative u' of $\mathcal{O}_u$ in the subgroup isomorphic to $SO_{\nu_1}(k) \times SO_{\nu_2+\nu_3}(k) \times \cdots \times SO_{\nu_{m-1}+\nu_m}(k) \times SO_{2\mu_1}(k) \times \cdots \times SO_{2\mu_l}(k)$ obtained using the embeddings in the proof of Theorem 3.4 and the embedding of $SO_{2d_1+1}(k) \times SO_{2d_2}(k)$ into $SO_{2(d_1+d_2)+1}(k)$ given by $\left(\begin{pmatrix} 1 & \alpha_1 & \beta_1 \\ \gamma_1 & A_1 & B_1 \\ \delta_1 & C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \mapsto$ $\begin{pmatrix} 1 & \alpha_1 & & \beta_1 & \\ \gamma_1 & A_1 & & B_1 & \\ & & A_2 & & B_2 \\ \delta_1 & C_1 & & D_1 & \\ & & C_2 & & D_2 \end{pmatrix}$. The component of u' in $SO_{\nu_i}(k)$ is regular therein, forcing $\nu_1 \leq 3$. Moreover, the argument used in Theorem 3.4 shows that $\mu_i \leq 2$ for every i. Conversely, for a unipotent class associated with $(2^m, 1^{2n+1-4m})$ or to $(3, 2^{2m}, 1^{2n-4m-2})$ the representative in [3, Theorem 12] is well-defined in odd characteristic and the corresponding B-orbit satisfies the condition in Theorem 2.2.

Let $g = su$ be the Jordan decomposition of a spherical element in G with s, $u \neq 1$. The argument in Theorem 3.4 can be adapted to the present situation, using the embedding ι. It follows that for any $\iota(t) \in \mathcal{O}_s$ we have $t \in Z(SO_{2n}(k))$. Therefore g is conjugate to $\rho_l \mathrm{v} = \mathrm{diag}(1, -I_{2n})\mathrm{v}$ for some v spherical unipotent in $G^\circ_{\rho_l} \cong SO_{2n}(k)$. We claim that the partition $\underline{\lambda}$ of $2n$ associated with v has no term equal to 3. Otherwise $\underline{\lambda} = (3, \underline{\lambda}', 1)$ for some partition $\underline{\lambda}'$ and g is conjugate

to a matrix of the form

$$y = \begin{pmatrix} 1 & 0 & 1 & & & & \\ 0 & -1 & 0 & & & & \\ 0 & 1 & -1 & & & & \\ & & & -A & & & -AX \\ 1 & 0 & \frac{1}{2} & & -1 & -1 & \\ 1 & 0 & \frac{1}{2} & & 0 & -1 & \\ & & & & & & -{}^tA^{-1} \end{pmatrix}$$

where $\begin{pmatrix} A & AX \\ & {}^tA^{-1} \end{pmatrix}$ is a unipotent matrix in $SO_{2n-4}(k)$ corresponding to $\underline{\lambda}'$ and A is upper triangular. The element y lies in $TX_{-\alpha_1}X_{-(\alpha_2+\cdots+\alpha_n)}B$ and it is contained in the Bruhat cell corresponding to $s_1 s_{\alpha_2+\cdots+\alpha_n}$ which is not an involution. Conversely, let $g = \rho_n u$ with u corresponding to $(2^{2m}, 1^{2n-4m})$. The representative of its class provided in [3, Theorem 21] is well defined in odd characteristic and it allows application of Theorem 2.2. $\square$

3.6 Type E_6

Theorem 3.6 *Let G be simply connected of type E_6. Then the spherical classes are either semisimple or unipotent. The semisimple ones are symmetric spaces, and, up to a central factor, they are represented by $p_1 = h_1(-1)h_4(-1)h_6(-1)$ and $p_{2,c} = h_1(c^2)h_2(c^3)h_3(c^4)h_4(c^6)h_5(c^5)h_6(c^4)$ for $c \in k$ with $c^3 \neq 1, 0$. The unipotent ones are those of type A_1, $2A_1$ and $3A_1$.*

Proof. Let $s \in T$ be spherical. By [11, Theorem 2.15] we may choose s so that G_s is generated by T and $X_{\pm\alpha}$ for α in a subsystem $\Phi(\Pi) \subset \Phi$ with basis a subset Π of $\Delta \cup \{-\beta_1\}$. By Theorem 2.2 we have $\dim \mathcal{O}_s \leq \ell(w_0) + \mathrm{rk}(1 - w_0)$ and a dimension counting shows that Π is one of the following subsets: $\Pi_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, -\beta_1\}$, $\Pi_2 = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, -\beta_1\}$ and $\Pi_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, -\beta_1\}$ of type $A_5 \times A_1$; $\Pi_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $\Pi_5 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, -\beta_1\}$, and $\Pi_6 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ of type $D_5 \times k^*$.

Let us put $H_i = \langle T, X_{\pm\alpha}, \alpha \in \Pi_i \rangle$ for $i = 1, \ldots, 6$. The sets Π_i for $i = 1, 2, 3$ are $\mathbb{R}$-bases for the span of Δ and one may find automorphisms of Φ mapping Π_i for $i = 2, 3$ to $\pm\Pi_1$. On the other hand, $\mathrm{Aut}(\Phi) = \{-w_0\} \ltimes W$ so any element s whose centralizer is H_2 or H_3 is $N(T)$-conjugate to an element whose centralizer is H_1. The elements s for which $G_s = H_1$ are $p_1 = h_1(-1)h_4(-1)h_6(-1)$ and

zp_1 for any $z \in Z(G)$. Conjugation by these elements is an involution so $\mathcal{O}_{p_1}$ is a symmetric space. This completes the analysis for Π_i with $i \leq 3$.

The subgroups H_4 and H_6 are $\dot{w}_0$-conjugate, so any element whose centralizer is H_4 is $N(T)$-conjugate to an element whose centralizer is H_6. Besides, the automorphism of Φ defined by $\alpha_1 \mapsto -\beta_1$, $\alpha_2 \mapsto \alpha_3$, $\alpha_3 \mapsto \alpha_2$, $\alpha_j \mapsto \alpha_j$ for $j = 4, 5, 6$ maps Π_4 onto Π_5 so H_5 is $N(T)$-conjugate to H_4 and any element whose centralizer is H_5 is $N(T)$-conjugate to an element whose centralizer is H_4. The elements whose centralizer is H_4 are $p_{2,c} = h_1(c^2)h_2(c^3)h_3(c^4)h_4(c^6)h_5(c^5)h_6(c^4)$ for $c \in k$ with $c^3 \neq 1, 0$. Multiplying c by a third root of unity yields the same element multiplied by a central one. Since $p_{2,-1}$ is an involution, $\mathcal{O}_{p_{2,c}}$ is a symmetric space by Remark 2.5. We claim that $p_{2,c}$ is not conjugate to $p_{2,d}$ for $c \neq d$. If they were G-conjugate they would be $N(T)$-conjugate by [19, §3.1] so there would be $\sigma \in W$ such that $\dot{\sigma} p_{2,c} \dot{\sigma}^{-1} = p_{2,d}$. Thus, σ would stabilize $\Phi(\Pi_4)$ and the restriction of σ to $\Phi(\Pi_4)$ is an automorphism. It is therefore of the form τw where τ acts an automorphism of the Dynkin diagram of type D_5 and w lies in the Weyl group W' of H_4, which is contained in W. Then σw^{-1} acts on Π_4 as τ. Besides, two automorphisms of Φ coinciding on Π_4 are equal. Indeed, α_6 is the only root α which is orthogonal to α_j for $j = 1, 2, 3, 4$ and for which $\langle \alpha, \alpha_5 \rangle = -1$. It follows that σw^{-1} is either the identity, when $\tau = 1$, or it is the automorphism mapping α_j to α_j for $j = 1, 3, 4$, interchanges α_2 and α_5 and maps α_6 to $-\beta_1$. However, one may verify that the second possibility cannot happen because such an automorphism is equal to $s_1 s_3 s_4 s_5 s_2 s_4 s_6 s_5 s_3 s_4 s_1 s_3 s_2 s_4 s_5 s_6 (-w_0)$, hence it does not lie in W. Therefore $\tau = 1$ and $\sigma = w \in W'$. It is not hard to verify that conjugation by the lift in $N(T)$ of an element in W' does not modify $p_{2,c}$ so these elements represent distinct classes.

Let $\mathcal{O}$ be a nontrivial spherical unipotent class. Then $\dim \mathcal{O} \leq \ell(w_0) + \mathrm{rk}(1 - w_0)$ by Theorem 2.2 so $\mathcal{O}$ is of type A_1, $2A_1$ or $3A_1$. Conversely, the arguments in [3, Theorem 13] apply in good characteristic and show that the listed orbits have a representative whose B-orbit satisfies the conditions of Theorem 2.2.

A dimension counting together with Lemma 2.6 shows that no class $\mathcal{O}_{su}$ with $s \notin Z(G)$ and $u \neq 1$ can be spherical. □

3.7 Type E_7

Theorem 3.7 *Let G be simply connected of type E_7. The spherical classes in G are either semisimple or unipotent. The semisimple ones are symmetric and are represented by $q_1 = h_2(\zeta)h_5(-\zeta)h_6(-1)h_7(\zeta)$ where ζ is a fixed primitive fourth root of 1; $q_2 = h_3(-1)h_5(-1)h_7(-1)$; zq_1, zq_2 for $z \in Z(G)$, and $q_{3,a} =$*

$h_1(a^2)h_2(a^3)h_3(a^4)h_4(a^6)h_5(a^5)h_6(a^4)h_7(a^3)$ with $a^2 \neq 1, 0$. The unipotent ones are those of type A_1, $2A_1$, $(3A_1)'$, $(3A_1)''$ and $4A_1$.

Proof. Let $s \in T$ be a spherical element. Proceeding as in Theorem 3.6 we may choose s so that G_s is generated by T and $X_{\pm\alpha}$ for $\alpha \in \Phi(\Pi)$ with Π one of the following subsets of $\Delta \cup \{-\beta_1\}$: $\Pi_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, -\beta_1\}$ of type A_7; $\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, -\beta_1\}$, $\Pi_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, -\beta_1\}$ of type $D_6 \times A_1$; and $\Pi_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ of type $E_6 \times k^*$. Let us put $H_i = \langle T, X_{\pm\alpha}, \alpha \in \Pi_i \rangle$.

There is only one element, up to a central one, whose centralizer is H_1 and this is $q_1 = h_2(\zeta)h_5(-\zeta)h_6(-1)h_7(\zeta)$ where ζ is a fixed primitive fourth root of 1. Since $q_1^2 = h_2(-1)h_5(-1)h_7(-1) \in Z(G)$ the corresponding class is symmetric. The root systems generated by Π_2 and Π_3 are mapped onto each other by automorphisms of Φ which therefore lie in W. Thus, each element whose centralizer is H_2 is $N(T)$-conjugate to one whose centralizer is H_3 and it is enough to look at Π_2. The elements whose centralizer is H_2 are $q_2 = h_3(-1)h_5(-1)h_7(-1)$ and zq_2 for $z \in Z(G)$. The corresponding classes are symmetric. The elements whose centralizer is H_4 are: $q_{3,a} = h_1(a^2)h_2(a^3)h_3(a^4)h_4(a^6)h_5(a^5)h_6(a^4)h_7(a^3)$ for $a^2 \neq 1, 0$. For ξ a primitive fourth root of unity we have $q_{3,\xi}^2 \in Z(G)$ hence all such classes are symmetric spaces. Multiplication of $q_{3,a}$ by the nontrivial central element gives $q_{3,-a}$. We claim that $q_{3,a}$ is never conjugate to $q_{3,b}$ for $a \neq b$. For, if they are G-conjugate, they are $N(T)$-conjugate, and there is a $\sigma \in W$ for which $\dot\sigma q_{3,a} \dot\sigma^{-1} = q_{3,b}$. Then σ preserves $\Phi(\Pi_4)$ and its restriction to it is an automorphism. As in the proof of Theorem 3.6 we see that there is some w in the Weyl group W' of H_4 for which the restriction to $\Phi(\Pi_4)$ of $\sigma w^{-1} \in W$ is an automorphism of the Dynkin diagram of type E_6. However, there is no automorphism of Φ whose restriction to E_6 is the nontrivial automorphism because there is no $\alpha \in \Phi$ which is orthogonal to α_j for $j = 2, 3, 4, 5, 6$ and for which $\langle \alpha, \alpha_1 \rangle = -1$. Therefore σw^{-1} is the identity on $\Phi(\Pi_4)$. By uniqueness of the extension of an automorphism from E_6 to E_7 we have $w = \sigma$. It is not hard to verify that conjugation by lifts in $N(T)$ of elements in W' preserves $q_{3,a}$.

Let $u \neq 1$ be a spherical unipotent element. Then $\dim \mathcal{O}_u \leq \dim B$ so $\mathcal{O}_u$ is either of type rA_1 for some r or of type A_2. In the latter case, u would be distinguished in a standard Levi subgroup of type A_2. Since in type A_n distinguished elements are regular, this case cannot occur by Remark 2.4. The arguments in [3, Theorem 13] apply also in good characteristic and they show that for all unipotent classes of type rA_1 there is a representative whose B-orbit satisfies the condition in Theorem 2.2.

We claim that there is no spherical element with Jordan decomposition $g = su$ with $s \notin Z(G)$ and $u \neq 1$. Indeed, $\mathcal{O}_s$ would be spherical and u would be spherical in G_s°. A dimensional argument shows that this could be possible only if $s \in \mathcal{O}_{q_2}$ and u is nontrivial only in the component of type A_1 in G_s. Then g would be conjugate to $q_2 x_{-\alpha_7}(1)$. Conjugation by $x_{-\alpha_6}(1)$ and Chevalley's commutator formula would give $q_2 x_{-\alpha_6}(a) x_{-\alpha_7}(1) x_{-\alpha_6-\alpha_7}(b) \in \mathcal{O}_g$ for some nonzero a, $b \in k$. Conjugating by a suitable element in $X_{-\alpha_6-\alpha_7}$ we could get rid of the term in $X_{-\alpha_6-\alpha_7}$ obtaining an element in $\mathcal{O}_g \cap B s_6 s_7 B$. By Theorem 2.3 the class $\mathcal{O}_g$ cannot be spherical. $\square$

3.8 Type E_8

Theorem 3.8 *Let G be of type E_8. The spherical classes are either semisimple or unipotent. The semisimple ones are symmetric and they are represented by $r_1 = h_2(-1)h_3(-1)$ and $r_2 = h_2(-1)h_5(-1)h_7(-1)$. The unipotent ones are those of type A_1, $2A_1$, $3A_1$ and $4A_1$.*

Proof. Let $s \in T$ be a spherical element. Since $\dim \mathcal{O}_s \leq \dim B$, up to conjugation in $N(T)$, the centralizer G_s is generated by T and by the $X_{\pm\alpha}$ for α in a subsystem generated either by $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, -\beta_1\}$ of type D_8 or by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, -\beta_1\}$ of type $E_7 \times A_1$. Then s is conjugate either to $r_1 = h_2(-1)h_3(-1)$ or to $r_2 = h_2(-1)h_5(-1)h_7(-1)$. Since $r_1^2 = r_2^2 = 1$ the corresponding classes are symmetric spaces.
Let $\mathcal{O}$ be a nontrivial spherical unipotent class. Then $\dim \mathcal{O} \leq \dim B$ so $\mathcal{O}$ is either of type rA_1 for some r, or it is of type A_2. The latter case is excluded as we did for G of type E_7. Conversely, the arguments in [3, Theorem 13] apply in good characteristic and they show that for each orbit of type rA_1 in G we may find a representative whose B-orbit satisfies the condition in Theorem 2.2.

We claim that there is no spherical element with Jordan decomposition $g = su$ with $s, u \neq 1$. Indeed, by dimensional reasons s would be conjugate to r_2 and u would lie in the component of type A_1 in G_s. In other words, we may assume $g = r_2 x_{-\beta_1}(1)$. Let $\gamma = \beta_1 - \alpha_8$. Conjugation of g by $\dot{s}_\gamma$ gives $t x_{-\alpha_8}(a) \in \mathcal{O}_g$ for some nonzero $a \in k$ and some $t \in T$. Since r_2 does not commute with $X_{\pm(\beta_1-\alpha_8-\alpha_7)}$ and $s_\gamma(\alpha_7 + \alpha_8 - \beta_1) = \alpha_7$ the element t does not commute with $X_{\pm\alpha_7}$. Since $s_\gamma(\alpha_7 + \alpha_8) = \alpha_7 + \alpha_8$ and r_2 does not commute with $X_{\pm(\alpha_7+\alpha_8)}$ the same holds for t. Then conjugation of $t x_{-\alpha_8}(a)$ by $x_{-\alpha_7}(1)$ would give $t x_{-\alpha_7}(b) x_{-\alpha_8}(a) x_{-\alpha_7-\alpha_8}(c) \in \mathcal{O}_g$ for some nonzero $b, c \in k$. Conjugation by a suitable element in $X_{-\alpha_7-\alpha_8}$ would yield an element in $\mathcal{O}_g \cap B s_7 s_8 B$ concluding the proof. $\square$

3.9 Type F_4

Theorem 3.9 *Let G be of type F_4. The spherical semisimple classes are symmetric spaces and they are represented by $f_1 = h_{\alpha_2}(-1)h_{\alpha_4}(-1)$ and $f_2 = h_{\alpha_3}(-1)$. The spherical unipotent ones are those of type $rA_1 + s\tilde{A}_1$ for $r, s \in \{0, 1\}$. Furthermore, there is a spherical class that is neither semisimple nor unipotent and it is represented by $f_2 x_{\beta_1}(1)$.*

Proof. Let $s \in T$ be a spherical semisimple element in G. A dimension counting shows that G_s is $N(T)$-conjugate to the subgroup generated by T and the root subgroups in a subsystem generated either by $\Pi_1 = \{-\beta_1, \alpha_2, \alpha_3, \alpha_4\}$ or by $\Pi_2 = \{\alpha_1, \alpha_2, \alpha_3, -\beta_1\}$. They correspond to $f_1 = h_{\alpha_2}(-1)h_{\alpha_4}(-1)$ and $f_2 = h_{\alpha_3}(-1)$, respectively.

Let $\mathcal{O}$ be a nontrivial spherical unipotent class in G. Then $\dim \mathcal{O} \leq \dim B$ so $\mathcal{O}$ is either of type $A_1, \tilde{A}_1$ or $A_1 + \tilde{A}_1$. Conversely, the arguments in [3, Theorem 13] hold in good characteristic and they show that Theorem 2.2 applies to all classes of type $rA_1 + s\tilde{A}_1$ in G.

Let $g = su$ be the Jordan decomposition of a spherical element with $s, u \neq 1$. Since $\dim \mathcal{O}_{f_1} = \dim B$ we might assume $s = f_2$. Besides, G_{f_2} is a reductive group of type B_4. A dimensional argument shows that u lies in the minimal unipotent class in G_{f_2} so we may assume $g = f_2 x_{2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4}(1)$. We have $\dim \mathcal{O}_g = \dim B$. The proof in [3, Theorem 23] contains an incorrect argument which we rectify here.

The element $f_2 = h_{\alpha_3}(-1)$ lies in the subgroup $G_1 = \langle X_{\pm\alpha_i},\ i = 2, 3, 4\rangle$ of type C_3. By looking at the centralizer of f_2 in G_1 we see that, up to an element in $Z(G_1)$, the conjugacy class of f_2 in G_1 corresponds to σ_1 with notation as in Theorem 3.3. By [3, Theorem 15], the class of σ_1 in G_1 has a representative in $s_4 s_{\alpha_2+2\alpha_3+\alpha_4} T$ when $k = \mathbb{C}$. The same matrix represents the class in good characteristic. Besides, G_1 centralizes $X_{\pm\beta_1}$, so $f_2 X_{-\beta_1}$ can be represented by an element $z \in s_4 s_{\alpha_2+2\alpha_3+\alpha_4} T X_{-\beta_1} \subset X_{\beta_1} w_0 s_2 T X_{\beta_1}$. Conjugating z by $\dot{s}_2 \dot{s}_1$ we obtain an element $z' \in B w_0 s_1 B \cap \mathcal{O}_g$. Thus, $w_{\mathcal{O}_g} \geq w_0 s_2$ and $w_{\mathcal{O}_g} \geq w_0 s_1$, forcing $w_0 = w_{\mathcal{O}}$. Then $\mathcal{O}_g$ has a representative whose B-orbit satisfies the condition in Theorem 2.2. $\square$

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